JOURNAL OF COMBINATORIAL THEORY, Series B 54, 178–188 (1992)

Generalized Local Colorings of Graphs

MIROSŁAW TRUSZCZYŃSKI

Department of Computer Science, University of Kentucky, Lexington, Kentucky 40506-0027

Communicated by the Editors

Received December 7, 1988

Let k be a fixed positive integer and let H be a graph with at least $k+1$ edges. A *local* (H, k)*-coloring* of a graph G is a coloring of the edges of G such that edges of no subgraph of G isomorphic to a subgraph of H are colored with more than k colors. In the paper we investigate properties of local (H, k)-colorings. We prove the Ramsey property for such colorings, establish conditions for the density property and the bipartite version of the Ramsey theorem to hold, and prove the induced variant of the Ramsey theorem with forbidden large cliques.

1. INTRODUCTION

In the paper we study a family of edge colorings of graphs. Let k be a fixed positive integer and let H be a graph with at least $k+1$ edges. A *local* (H, k)*-coloring* (or simply, an (H, k)*-coloring*) of a graph G is a coloring of the edges of G that satisfies the following property:

> edges of no subgraph of G isomorphic to a subgraph of H are colored with more than k colors.

The idea is to allow an arbitrary number of colors in a coloring as long as "locally" at most k colors are used. Local (H, k)-colorings generalize *local k-colorings* introduced in [4] as edge colorings of a graph such that the edges incident to each vertex are colored with at most k different colors. It is evident that local k-colorings are exactly local $(K_{1,k+1}, k)$-colorings. In addition, directly from the definition the following claim follows.

CLAIM 1.1. *If H_1 is a subgraph of H_2 and H_1 has at least $k+1$ edges, then for every G, an (H_2, k)-coloring of G is also an (H_1, k)-coloring of G.*

Local k-colorings were introduced as a generalization of usual k-colorings, i.e., edge colorings using at most k colors. In [4, 3, 7] properties of local k-colorings were studied. Even though the number of colors in a local k-coloring may be much larger than k, in many aspects local

0095-8956/92 $3.00

k-colorings behave similarly to k-colorings. Many similarities exist, in particular, in the area of Ramsey Theory. For example, it is true [4] that for every graph G, the local Ramsey number $r^k_{\mathrm{loc}}(G)$ is well defined; i.e., there is the least integer t such that for every local k-coloring of the complete graph on t vertices, there exists a monochromatic subgraph isomorphic to G. In addition, for many graphs G the usual Ramsey number $r^k(G)$ and the local Ramsey number $r^k_{\mathrm{loc}}(G)$ are equal or differ by 1. A general result in [7] states that for every integer $k \geqslant 1$ there is a constant c_k such that for all connected graphs G, $r^k_{\mathrm{loc}}(G) \leqslant c_k r^k(G)$. Since for any graph G, $r^k(G) \leqslant r^k_{\mathrm{loc}}(G)$, for connected graphs usual and local Ramsey numbers are within a constant factor from each other.

Other properties of local k-colorings were studied in [3]. It is shown there that as for usual k-colorings, the density property, the bipartite Ramsey theorem, and the induced Ramsey theorem can be proved for local k-colorings. Among the most notable differences between the two types of colorings are the fact that for some families of disconnected graphs the ratio $r^k_{\mathrm{loc}}(G)/r^k(G)$ cannot be bounded (see [4] for details), and that for all n there is an n-chromatic graph with a local 2-coloring such that all monochromatic subgraphs are bipartite (see [3]), whereas if a graph has a 2-coloring with that property, it must be 4-chromatic.

In this paper we study properties of local (H, k)-colorings. There are several similarities but also more differences between local (H, k)-colorings and k-colorings. Let $r^{(H,k)}(G)$ be the least integer t such that for every (H, k)-coloring of the complete graph with t vertices, there exists a monochromatic subgraph isomorphic to G. We show that $r^{(H,k)}(G)$ is well defined for all graphs G if and only if H contains a forest with $k+1$ edges (Section 2). We show also that for such graphs H, the induced version of the Ramsey theorem remains true (Section 4). On the other hand, the density property and the bipartite Ramsey theorem for (H, k)-colorings are not true, in general, and hold only if H contains a forest of stars of size $k+1$ (Section 3).

In the paper we use the following notation. If F is a forest of size $k+1$ we abbreviate (F, k)-coloring to F-coloring, and $r^{(F,k)}(G)$ to $r^F(G)$. We write K_X to denote the complete graph with vertex set X (K_t, if $X = \{1, ..., t\}$). For an edge coloring ϕ of a graph G and for $X \subseteq V(G)$, we denote by $\phi \mid X$ the restriction of ϕ to the set of edges of the subgraph of G induced by X. A subgraph H of an edge-colored graph is called a *rainbow* if no two edges of H have the same color.

Finally, let us note that in the case when $k=1$, (H, k)-colorings are easy to describe. We have the following simple result.

PROPOSITION 1.2. *Let G and H be graphs and let ϕ be an $(H, 1)$-coloring of G.*

(1) *If H contains $K_{1,2}$, then every connected component of G is monochromatic (under ϕ).*

(2) *If H contains $2K_2$, and G has at least 3 independent edges, then ϕ is a 1-coloring (G is monochromatic).*

2. Ramsey Theorem for (H, k)-Colorings

Our main results in this section establish necessary and sufficient conditions for $r^{(H,k)}(G)$ to be well defined for every graph G. In the proof we need several auxiliary lemmas.

LEMMA 2.1. *Let F be a forest with $k+1$ edges. For every finite set X and every F-coloring ϕ of K_X there is $Y \subseteq X$ such that*

(i) *$\phi \mid Y$ is a local $3k$-coloring of K_Y,*

(ii) $|Y| \geqslant |X|/(2k+1)$.

Proof. Let A be the set of all vertices of X such that $x \in A$ if and only if the edges incident to x are colored with at most $3k$ colors. If $A = \emptyset$ then set $X_1 = X$. Otherwise, set $X_1 = A$. Suppose sets $X_1, \ldots, X_i$ have been defined. Put $X' = X \setminus (X_1 \cup \cdots \cup X_i)$. If $X' = \emptyset$, stop the construction. Otherwise, let A to be the subset of X' such that $x \in A$ if and only if $x \in X'$ and the edges of $K_{X'}$ incident to x are colored with at most $3k$ colors. If $A = \emptyset$, then define $X_{i+1} = X'$, if $A \neq \emptyset$, define $X_{i+1} = A$. Suppose the process stops after sets $X_1, \ldots, X_p$ have been defined. Clearly, $\{X_1, \ldots, X_p\}$ is a partition of X. Let us define $Z_i = X_i \cup \cdots \cup X_p$, for $1 \leqslant i \leqslant p$.

(1) It follows from the definition of the X_i's that if $x \in Z_i$, $i > 1$, then the edges of $K_{Z_{i-1}}$ incident to x are colored with at least $3k+1$ colors.

(2) Also directly from the definition, it follows that for $i = 1, 2, \ldots, p-1$, $\phi \mid X_i$ is a local $3k$-coloring of K_{X_i}. Moreover, either $\phi \mid X_p$ is a local $3k$-coloring of K_{X_p}, or each vertex of K_{X_p} is incident (in K_{X_p}) to edges colored with at least $3k+1$ colors. Suppose that the latter possibility holds. Then, it is easy to construct in K_{X_p} a rainbow copy of F (or the subgraph of F obtained by removing from F all isolated vertices), which is a contradiction, as ϕ is an F-coloring. So, for every $1 \leqslant i \leqslant p$, $\phi \mid X_i$ is a local $3k$-coloring.

(3) Assume that $p \geqslant 2k+2$. We will show that for every forest T, without isolated vertices, and with $|E(T)| = i \leqslant k+1$, there is a rainbow copy of T in $K_{Z_{2k+3-2i}}$. We proceed by induction on i. For $i = 1$ the statement is trivial. So, consider a forest T without isolated vertices, and with $|E(T)| = i$, $1 < i \leqslant k+1$. Let u be a pendant vertex in T, and let v be the

neighbor of u in T. If uv is an isolated edge in T, set $T' = T - u - v$. Otherwise, set $T' = T - u$. By the induction hypothesis, there is a rainbow copy of T', say T'', in $K_{Z_{2k+3-2(i-1)}}$. Let v'' be the vertex of T'' corresponding to v in T' (if $v \in V(T')$) or any vertex of $X_{2k+3-2i+1}$ (if $v \notin V(T')$). There are $3k+1$ differently colored edges incident to v'' in $K_{Z_{2k+3-2i}}$ (by (1)). Hence, it is possible to choose among these edges an edge e, colored differently from all edges of T'', and whose other endvertex is not in $V(T'')$. Clearly, $T'' \cup \{e\}$ is a rainbow copy of T in $K_{Z_{2k+3-2i}}$.

(4) Let T be the forest obtained from F by removing all its isolated vertices. By (3), there is a rainbow copy of T in K_X. But as ϕ is an F-coloring, it is a T-coloring, a contradiction, so that $p \leqslant 2k+1$. Since $\{X_1, ..., X_p\}$ is a partition of X, we have $|X_i| \geqslant |X|/(2k+1)$, for some $1 \leqslant i \leqslant p$. From (2), $\phi \mid X_i$ is a local $3k$-coloring. Hence $Y = X_i$ satisfies the assertion. ∎

The proof of the next lemma is similar.

LEMMA 2.2. *Let F be a forest with $k+1$ edges. For every finite set X and every edge coloring ϕ of K_X that is both an F-coloring and a local $3k$-coloring, there is a subset Y of X satisfying*

(i) *$\phi \mid Y$ is a local k-coloring of K_Y,*

(ii) $|Y| \geqslant |X|/((2+6k^2)(2k+1))$.

Proof. We first introduce some notation. We call a color *rare at x in K_X* if it appears on at most $2k$ edges incident in K_X to x. Otherwise, a color is called *common at x in K_X*. A vertex x is *irregular* if there are at most k common colors among colors used to color edges incident to x, otherwise, x is called *regular*.

Let A consist of all irregular vertices in K_X. If $A = 0$ then set $X_1 = X$ otherwise, set $X_1 = A$. Suppose sets $X_1, ..., X_i$ have been defined and let $X' = X \backslash (X_1 \cup \cdots \cup X_i)$. If $X' = \varnothing$ then stop the construction. Otherwise, let A be the set of all irregular vertices in $K_{X'}$. If $A = \varnothing$, then let $X_{i+1} = X'$, and stop. If $A \neq \varnothing$, define $X_{i+1} = A$, and continue. Let us suppose that the process terminates after sets $X_1, ..., X_p$ are constructed. Let us define $Z_i = X_i \cup \cdots \cup X_p$, for $1 \leqslant i \leqslant p$.

(1) Observe that if $x \in Z_i$, $i > 1$, then x is regular in $K_{Z_{i-1}}$.

(2) Observe also that for every $1 \leqslant i \leqslant p-1$, each $x \in X_i$ is irregular in K_{X_i}. Moreover, either each $x \in X_p$ is irregular in K_{X_p}, or each $x \in X_p$ is regular in K_{X_p}. If this latter case holds, then a rainbow copy of F can be constructed (using edges of common colors only). As this contradicts our assumptions, we obtain that also for $i = p$ each $x \in X_i$ is irregular in K_{X_i}.

(3) Assume that $p \geqslant 2k+2$. We claim (as in step (3) of the proof of Lemma 2.1) that for every forest T, without isolated vertices, and with $|E(T)| = i \leqslant k+1$, there is a rainbow copy of T in $K_{Z_{2k+3-2i}}$. To prove this we proceed by induction on i. For $i=1$ the statement is trivial. So, consider a forest T without isolated vertices, and with $|E(T)| = i$, $1 < i \leqslant k+1$. In the induction step, we use the notation from step (3) of Lemma 2.1. In order to find an edge that can be added to the rainbow forest T'' at vertex v'', we choose a color common at v'' that is different from all colors used on T'' (it is possible; by (1) v'' is regular in $K_{Z_{2k+3-2i}}$ hence, there are at least $k+1$ common colors at v''). Then. from among $2k+1$ edges of that color incident to v'' in $K_{Z_{2k+3-2i}}$, we choose an edge whose other endvertex does not belong to T'', and we add this edge to T'' to get a rainbow copy of T, contained in $K_{Z_{2k+3-2i}}$. This completes the proof of our claim. Consequently, a rainbow copy of F (or its subgraph obtained from F by removing all isolated vertices) can be found in K_X, a contradiction. Hence, $p \leqslant 2k+1$.

(4) From (3) we have that for some X_i, $1 \leqslant i \leqslant p$, $|X_i| \geqslant |X|/(2k+1)$, and by (2), all vertices in X_i are irregular in K_{X_i}. Let u be a new color, not used by ϕ. For each vertex $x \in X_i$, recolor with u all the edges of K_{X_i} incident to x and colored (by ϕ) with a color that is rare at x. Let G be the subgraph of K_{X_i} spanned by the edges that were not recolored. Since at each vertex there are at most $3k$ colors (ϕ is a local $3k$-coloring), at most $6k^2$ edges incident to x are recolored. Thus, $\delta(G) \geqslant |X_i| - 1 - 6k^2$ ($\delta(G)$ denotes the minimum degree). Each edge e of G incident to x is colored with a color common at x. Since there are at most k colors common at x, $\phi \mid V(G)$ is a local k-coloring. Set $r = \lfloor |X_i|/(2+6k^2) \rfloor + 1$. If $r > 1$ then,

$$\delta(G) \geqslant |X_i| - 1 - 6k^2 > (1 - 1/(r-1))\, |X_i|.$$

Thus, by Turán's Theorem (see [1, Corollary 1.3, p. 295]) there is $Y \subseteq X_i$ such that $|Y| = r$ and the subgraph of G induced by Y is complete. If $r=1$, the existence of such Y is obvious. Clearly, Y satisfies (i) and (ii) of the assertion. ∎

The next result follows directly from Lemmas 2.1 and 2.2.

LEMMA 2.3. *Let F be a forest with $k+1$ edges. For every finite set X and every F-coloring of K_X there is $Y \subseteq X$ satisfying*:

(i) *$\phi \mid Y$ is a local k-coloring of K_Y*,

(ii) $|Y| \geqslant |X|/((2+6k^2)(2k+1)^2)$.

We are ready to prove the main result of the section giving conditions for the existence of Ramsey numbers $r^{(H,k)}(G)$.

THEOREM 2.4. *Let H be a graph with at least $k+1$ edges. The Ramsey number $r^{(H,k)}(G)$ is well defined for every graph G if and only if H contains a forest with $k+1$ edges. Moreover, in such case, $r^{(H,k)}(G) \leqslant (2+6k^2)(2k+1)^2 r^k_{\mathrm{loc}}(G)$.*

Proof. Let F be a forest with $k+1$ edges contained in H. Clearly, every (H,k)-coloring is an F-coloring (by Claim 1.1). Hence, sufficiency and the "moreover" part follow from Lemma 2.3. To prove necessity, let us suppose that $X=\{1, 2, ..., n\}$. Consider a coloring ϕ_n of K_X such that $\phi_n(\{i, j\}) = \min(i, j)$, for every edge ij of K_X. Suppose that a set A of $k+1$ edges is colored with $k+1$ different colors. It is easy to see that A spans a forest. So, if H does not contain a forest with $k+1$ edges, then ϕ_n is an (H,k)-coloring. However, under colorings ϕ_n, the only monochromatic graphs are stars. ∎

Several problems seem to be worth investigating. First, it is easy to see that for $k \geqslant 2$, every (K_{2k}, k)-coloring of K_n is a k-coloring. Thus, trivially, for every graph and for $k \geqslant 2$, $r^k(G) = r^{(K_{2k},k)}(G)$. The following question arises: What is the smallest integer $m \geqslant k+2$ such that for every graph G, $r^k(G) = r^{(K_m,k)}(G)$? A similar question can be asked if we restrict to connected graphs only. For that case we have the following result.

THEOREM 2.5. *Let $k \geqslant 1$ and $m = \lceil 3k/2 \rceil + 1$. Then, for every connected graph G, $r^{(K_m,k)}(G) = r^k(G)$.*

Proof. Let G be a connected graph. It suffices to prove that $r^{(K_m,k)}(G) \leqslant r^k(G)$. So, let $n = r^k(G)$ and let ϕ be any (K_m, k)-coloring of K_n. Suppose also that no monochromatic copy of G can be found. Let c_1 and c_2 be two colors of ϕ such that no vertex is incident to edges of both colors (if such colors exist). Let ϕ' be the coloring obtained from ϕ by recoloring with c_2 all edges colored (by ϕ) with c_1. Clearly, ϕ' is a (K_m, k)-coloring, as well. Moreover, as G is connected, no monochromatic (with respect to ϕ') copy of G exists. Repeat the above recoloring procedure until we obtain a coloring ψ such that for every two colors c_1 and c_2 of ψ there is a vertex incident to edges of both colors. By the construction, ψ is a (K_m, k)-coloring and no monochromatic (with respect to ψ) copy of G exists.

As $n = r^k(G)$, ψ uses at least $k+1$ colors. If $k+1$ is even, choose any $k+1$ colors $c_1, ..., c_{k+1}$. If $k+1$ is odd, first find three colors a, b, and c such that for some four vertices w, x, y, and z each of a, b, and c is used to color an edge in $K_{\{w,x,y,z\}}$ and then choose any other $k-2$ colors $c_1, ..., c_{k-2}$. For $i = 1, 2, ..., (k+1)/2$ if $k+1$ is even (for $i = 1, 2, ..., (k-2)/2$, if $k+1$ is odd) choose vertices x_i, y_i, and z_i so that $\psi(x_i y_i) = c_{2i-1}$ and $\psi(y_i z_i) = c_{2i}$. Let X be the set of all selected vertices.

We have $|X| \leqslant \lceil 3k/2 \rceil + 1 = m$ and there are at least $k+1$ colors of ψ on the edges of K_X. Hence, ψ is not a (K_m, k)-coloring, a contradiction. ∎

Another, even more general question is the following one: What are the minimal graphs H containing a forest with $k+1$ edges and such that for every graph G, $r^{(H,k)}(G) = r^k(G)$? If $k = 2$, a triangle with one pendant edge attached to it has all the properties.

The case when H is a forest with $k+1$ edges seems to be particularly interesting, as forests are the minimal graphs for which local Ramsey numbers are defined. We have the following two simple results.

THEOREM 2.6. *Let $k \geqslant 2$. For every graph G, $r^{(k+1)K_2}(G) \leqslant r^k(G) + 2k$.*

Proof. Set $n = r^k(G) + 2k$ and let $X = \{1, 2, ..., n\}$. Let ϕ be a $(k+1)$ K_2-coloring of K_X. We need to show that a monochromatic copy of G exists. This is clearly the case if ϕ uses no more than k colors. So, suppose otherwise. Choose any edge wx; say its color is 1. Consider $K_{X \setminus \{w,x\}}$. If $\phi \mid X \setminus \{w, x\}$ uses no more than k colors, we are done as $|X \setminus \{w, x\}| \geqslant r^k(G)$. If $\phi \mid X \setminus \{w, x\}$ uses more than k-colors, choose an edge yz of color other than 1, say $\phi(yz) = 2$. Consider $K_{X \setminus \{w, x, y, z\}}$ and repeat the process. Since ϕ is a $(k+1)$ K_2-coloring, after at most k steps a set $Y \subseteq X$ is obtained with $|Y| \geqslant r^k(G)$ and such that $\phi \mid Y$ uses at most k colors. Thus, a monochromatic copy of G exists in K_Y. ∎

Let D_k $(k \geqslant 2)$ be a tree obtained from a star with k edges by attaching a new pendant edge to one of non-central vertices of the star. We have the following fact.

PROPOSITION 2.7. *For every graph $G \neq K_{1,2}$, $r^{D_k}(G) \leqslant r^k_{\text{loc}}(G)$.*

Proof. It is easy to see that if $|X| \neq k+2$ then every D_k-coloring of K_X is a local k-coloring. Thus, if $r^k_{\text{loc}}(G) \neq k+2$, then $r^{D_k}(G) \leqslant r^k_{\text{loc}}(G)$. Suppose now that $r^k_{\text{loc}}(G) = k+2$. There is a k-coloring of K_{k+2} such that each monochromatic set is a path. There is also a k-coloring of K_{k+2} such that each monochromatic set is a star or a triangle. Thus, if $r^k_{\text{loc}}(G) = k+2$, then $G = K_{1,2}$. ∎

The following problems arise:

(1) Let F be a forest of $k+1$ edges. How does the Ramsey number $r^F(G)$ depend on the structure (diameter of components) of F?

(2) Is the following true: For every forest F with $k+1$ edges and for every (sufficiently large) graph G, $r^F(G) \leqslant r^k_{\text{loc}}(G)$?

3. The Density Property and the Bipartite Ramsey Theorem

A property of edge colorings that found many important applications is the density property [5]. For a local k-coloring (in particular, a k-coloring) of a graph G, it states that there exists a monochromatic subgraph H of G such that $d^*(H) \geq d^*(G)/k$ [5, 3], where $d^*(G)$ denotes the average degree of a vertex in G, i.e., $d^*(G) = 2\,|E(G)|/|V(G)|$. Although local (H, k)-colorings share many common properties with k-colorings and local k-colorings, they behave differently with respect to the density property.

Let us consider the coloring ϕ of a complete bipartite graph $K_{n,n}$ with color classes $\{x_0, ..., x_{n-1}\}$, $\{y_0, ..., y_{n-1}\}$, where all the edges incident with x_i are colored with color i. Let F be a forest. Clearly, if F is not a forest of stars, ϕ is an F-coloring of $K_{n,n}$ (as F must contain two edges incident to some vertex x_i and, consequently, colored with i). Since the density of $K_{n,m}$ is n, and the density of every monochromatic subgraph is less than 2, the density property does not hold. When F is a forest of stars we have the following density theorem. (Note that it gives a weaker lower bound for the density of an existing monochromatic subgraph than the density theorem for local k-colorings, i.e., for the case when F is a star.)

THEOREM 3.1. *Let F be a forest of stars with $k+1$ edges. If a graph G is F-colored then there is a monochromatic subgraph H of G such that $d^*(H) \geq d^*(G)/4k$.*

Proof. Let G_i be the subgraph of G spanned by all edges of G colored with i, and let $d_i(x)$ be the number of edges of color i incident to vertex x. Clearly, $d(x) = \sum_i d_i(x)$, and $\sum_{x \in V(G)} d(x) = d^*(G)\,|V(G)|$. Thus,

$$d^*(G)\,|V(G)| = \sum_i \sum_{x \in V(G)} d_i(x) = \sum_i d^*(G_i)\,|V(G_i)|.$$

Let p be the color with the property that $d_p^*(G)$ is maximum. Then

$$d^*(G)\,|V(G)| \leq d^*(G_p) \sum_i |V(G_i)|.$$

Let q be the number of vertices of G that have their incident edges colored with at least $3k+1$ colors. Then $q \leq k$ (otherwise a rainbow F could be constructed). Hence,

$$\sum_i |V(G_i)| \leq q(|V(G)| - 1) + 3k(|V(G)| - q) \leq 4k\,|V(G)|.$$

Consequently, $d^*(G_p) \geq d^*(G)/4k$. ∎

In the case of local k-colorings, the density theorem implies the bipartite Ramsey theorem (see [5, 3]). Similarly, our density result implies the bipartite Ramsey theorem for the class of F-colorings, where F is a forest of stars.

THEOREM 3.2. *Let F be a forest of stars with $k+1$ edges. For every m there exists n so that if $K_{n,n}$ is F-colored then there exists a monochromatic copy of $K_{m,m}$.*

The bipartite Ramsey theorem does not hold in general, as is showed by the coloring ϕ described at the beginning of this section. A similar construction shows that the following result for k-colorings does not hold, in general, for F-colorings.

THEOREM 3.3 [5, Lemma, p. 103; 3]. *For every bipartite graph H and every k, there exists a bipartite graph G such that when G is (locally) k-colored, then it contains a monochromatic copy of H as an induced subgraph of G.*

This result is the key in the proof of the induced version of Ramsey theorem with forbidden large complete subgraphs (see [6] for the case of k-colorings, and [3] for the case of local k-colorings). The induced version of Ramsey theorem with forbidden large comple subgraphs holds for local (H, k)-colorings, but the proof is different. We present the details in the next section.

4. Induced Version of the Ramsey Theorem for Local (H, k)-Colorings

The purpose of this section is to give a proof of the following strengthening of the Ramsey theorem for local colorings.

THEOREM 4.1. *Let F be a forest with $k+1$ edges. For every graph G there exists a graph H such that $cl(H) = cl(G)$ and in every F-coloring of H there is a monochromatic induced subgraph isomorphic to G.*

In the proof we use two results that establish a similar statement but for a different class of colorings.

THEOREM 4.2 (Folkman [2]). *For every $r \geqslant 1$ and for every graph G there exists a graph H such that $cl(H) = cl(G)$ and in every coloring of the vertices of H with r colors there is a monochromatic induced subgraph isomorphic to G.*

THEOREM 4.3 (Gyárfás *et al.* [3], Nešetřil and Rödl [6] for r-colorings). *For every $r \geqslant 1$ and for every graph G there exists a graph H such that $cl(H) = cl(G)$ and in every local r-coloring of the edges of H there is a monochromatic induced subgraph isomorphic to G.*

Proof of Theorem 4.1. We proceed by induction on $|E(F)| = k+1$. If $|E(F)| = 2$ $(k=1)$, the assertion follows from Proposition 1.2. So, assume that $|E(F)| = k+1$, where $k > 1$. Let $x \in V(F)$ be a vertex of degree 1 (in F) and let $F' = F - x$. Let $y \in V(F)$ be the neighbor of x in F.

(1) Define H_1 to be a graph such that $cl(H_1) = cl(G)$ and for every F'-coloring of H_1 there is a monochromatic induced subgraph isomorphic to G. (The existence of such an H_1 follows by the induction hypothesis.) Further, let H_2 be a graph such that $cl(H_2) = cl(G)$, and for every local k-coloring of H_2 there is a monochromatic induced subgraph isomorphic to G. (The existence of such a graph follows from Theorem 4.3.) Finally, let H_3 be a graph such that $cl(H_3) = cl(H_2)$ $(= cl(G))$ and, for every coloring of the vertex set of H_3 with $|V(H_1)|$ colors there is a monochromatic induced subgraph isomorphic to H_2. (The existence of such a graph follows from Theorem 4.2.) Define H as the cartesian product of H_3 and H_1.

(2) It follows from the construction that $cl(H) = cl(G)$. It remains to show that for every F-coloring of H there is a monochromatic induced subgraph isomorphic to G. First, we introduce some notation. For $v \in V(H_3)$, by (v, H_1) we mean the subgraph of $H (= H_3 \times H_1)$ induced by $\{v\} \times V(H_1)$. For $v \in V(H_1)$, the graph (H_3, v) is defined analogously. Clearly, (v, H_1) is isomorphic to H_1 and (H_3, v) is isomorphic to H_3.

(3) Let ϕ be an F-coloring of H. If there is $v \in V(H_3)$ such that $\phi\,|(v, H_1)$ is an F'-coloring of (v, H_1), then a monochromatic induced copy of G exists in (v, H_1) and consequently in H, as (v, H_1) is an induced subgraph of H. So, suppose that for every $v \in V(H_3)$, ψ_v is an embedding of F' into (v, H_1) such that $\psi_v(F')$ is isomorphic to F' and has all edges of different colors.

(4) If there are $k+1$ edges of different colors incident to vertex $(v, \psi_v(y))$ in the graph $(H_3, \psi_v(y))$, then a rainbow copy of F can be found in H. (Extend $\psi_v(F')$ by attaching to $\psi_v(y)$ an edge of $(H_3, \psi_v(y))$ of color different from all colors of the edges of $\psi_v(F')$.) Thus, for every $v \in V(H_3)$ there are at most k edges of different colors incident to $(v, \psi_v(y))$ in $(H_3, \psi_v(y))$.

(5) Color each vertex of H_3 with $\psi_v(y)$. The number of colors used is at most $|V(H_1)|$ hence, from the definition of H_3 it follows that there is in H_3 an induced subgraph H_2', isomorphic to H_2 and with all vertices

colored with the same element, say $z \in V(H_1)$. Thus, by (4), $\phi \mid V(H_2') \times \{z\}$ is a local k-coloring of $H_2' \times \{z\}$. Consequently, $H_2' \times \{z\}$ contains an induced monochromatic subgraph isomorphic to G. Since $H_2' \times \{z\}$ is an induced subgraph of H it follows that there is a monochromatic induced copy of G in H. ∎

The following corollary follows directly from Claim 1.1

COROLLARY 4.4. *Let F be a graph containing a forest with $k+1$ edges. For every graph G there exists a graph H such that $cl(H) = cl(G)$ and in every (F, k)-coloring of H there is a monochromatic induced subgraph isomorphic to G.*

REFERENCES

1. B. BOLLOBÁS, "Extremal Graph Theory," Academic Press, London, 1978.
2. J. FOLKMAN, Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM J. Appl. Math.* **18** (1970), 19–24.
3. A. GYÁRFÁS, J. LEHEL, J. NEŠETŘIL, V. RÖDL, R. H. SCHELP, AND ZS. TUZA, Local k-colorings of graphs and hypergraphs, *J. Combin. Theory Ser. B* **43** (1987), 127–139.
4. A. GYÁRFÁS, J. LEHEL, R. H. SCHELP, AND ZS. TUZA, Ramsey numbers for local colorings, *Graphs Combin.* **3** (1987), 267–277.
5. R. L. GRAHAM, B. L. ROTHSCHILD, AND J. H. SPENCER, "Ramsey Theory," Wiley, New York, 1980.
6. J. NEŠETŘIL AND V. RÖDL, The Ramsey property for graphs with forbidden complete subgraphs, *J. Combin Theory Ser. B* **20** (1976), 243–249.
7. M. TRUSZCZYŃSKI AND ZS. TUZA, Linear upper bounds for local Ramsey numbers, *Graphs Combin.* **3** (1987), 67–73.